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# Breathing relativistic rotators and fundamental dynamical systems

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**Abstract**

According to the Staruszkiewicz definition, a relativistic dynamical system is said to be fundamental if its Casimir invariants are parameters, not constants of motion. This property is the classical counterpart of quantum irreducibility idea, which makes fundamental dynamical systems particularly interesting. An example of a fundamental dynamical system is provided by the fundamental relativistic rotator known also as the $(m, s)$ particle. Recently, however, it turns out that the rotator is defective as a dynamical system. Therefore, as a first step toward finding a well behaving fundamental dynamical system, a class of breathing rotators, being a natural extension of the smallest class containing the fundamental relativistic rotator, is considered. A breathing rotator is a relativistic dynamical system consisting of its worldline and a single null vector. Surprisingly, fundamental dynamical systems from this extended class also turn out to be defective.

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## 1. Introduction

This work concerns a class of breathing rotators which are relativistic dynamical systems consisting of a single null vector $k$ associated with position $x$. It provides the simplest extension of a class of rotators considered by Staruszkiewicz [1]. As he noted, a rigid body of Hanson and Regge [3], that is, a tetrad associated with a position and moving in accordance with some relativistically invariant laws of motion, could be equivalently characterized as a dynamical system described by position and three null directions. As a particular realization of such a body, Staruszkiewicz considered relativistic rotators, that is, rigid bodies characterized by a single null direction associated with a position.

To reduce, with a single clear-cut physical idea, the enormous variety of relativistically invariant actions possible for such systems as above, one can use the notion of a fundamental dynamical system introduced by Staruszkiewicz [1] and which can be rephrased as follows:

> *A dynamical system described by a relativistically invariant action is said to be fundamental if its Casimir invariants of the Poincaré group are parameters with fixed numerical values, rather than arbitrary constants of motion.*

By applying this definition, one obtains two independent constraints that must be satisfied by the action of a dynamical system. These two constraints are referred to here as *fundamental conditions*.

Representations of relativistic quantum mechanical systems are labeled by the numerical values of Casimir invariants of different symmetry groups, among which the most important is the Poincaré group [2]. Similarly, the fundamental conditions provide classical counterpart of quantum irreducibility when a relativistic classical fundamental system can be indicated uniquely by fixing the numerical values of its mass and of its intrinsic spin, independently of the state of motion.

It is obvious that this 'classical irreducibility idea' already suffices to fully justify the necessity of research on relativistic fundamental classical systems. Another reason is that a subset of fundamental systems whose motion is periodic could be used as ideal classical clocks which are purely mathematical constructions experiencing no fatigue and friction [1]. The Staruszkiewicz fundamental relativistic rotator [1], which is uniquely determined by fundamental conditions, is indeed an ideal clock. Its motion is periodic and its mass and intrinsic spin are unaffected even when it interacts with an external electromagnetic field like a charged particle—this interesting property was confirmed by the authors of [4]. In fact, construction of such a clock was the main motivation of [1]. It must be stressed here that this fundamental rotator was first found by Kuzenko *et al* [6], and even its quantized version was considered therein. Staruszkiewicz, starting from different premises, independently rediscovered it 15 years later (cf comments in [4]).

Surprisingly, it has recently been shown [5] that the fundamental relativistic rotator is defective as a dynamical system—accelerations are not uniquely determined from velocities and positions. This deficiency already disappears by an arbitrary small deformation of action of the rotator, however, at the cost of losing the feature of being fundamental. This shows that the defectiveness is not caused by the number of degrees of freedom but is inherent to the fundamental rotator.

One can hypothesize that a fundamental dynamical system must be sufficiently complex in order to ensure well-posedness of the Cauchy problem. In this respect, by considering breathing rotators, it is examined whether inclusion of an additional degree of freedom would suffice to eliminate this degeneracy, whilst not violating the fundamental conditions. Unfortunately, as will be shown later, breathing rotators are still too simple dynamical systems to be fundamental.

## 2. Hessian determinant and functional dependence of Casimir invariants

There are four nonzero Poincaré invariants formed from $x$, $k$ and their first derivatives: $\dot{x}\dot{x}$, $k\dot{x}$, $\dot{k}\dot{x}$, $\dot{k}\dot{k}$. However, not every combination of the invariants is suitable for a relativistically invariant Hamilton's action. Such an action must be reparametrization invariant. In addition, in order to reduce the number of free parameters to a minimum, this action is assumed to be independent of the physical dimension of $k$. Consequently, satisfying these requirements, the most general action for breathing rotators reads

$$-m\int\sqrt{\dot{x}\dot{x}}\,F(\mathcal{P},\mathcal{Q})\,\mathrm{d}\tau,\qquad \mathcal{P}=\ell\frac{\dot{k}\dot{x}}{k\dot{x}\sqrt{\dot{x}\dot{x}}},\qquad \mathcal{Q}=-\ell^2\frac{\dot{k}\dot{k}}{(k\dot{x})^2}. \tag{2.1}$$

Dimensional constants $m$ (mass) and $\ell$ (length) are the only parameters of the model. The momenta canonically conjugated with $x$ and $k$ are $P\equiv-\frac{\partial L}{\partial\dot{x}}$ and $\Pi\equiv-\frac{\partial L}{\partial\dot{k}}$, or explicitly,

$$P=m\left((F-\mathcal{P}F_{,\mathcal{P}})u-(2\mathcal{Q}F_{,\mathcal{Q}}+\mathcal{P}F_{,\mathcal{P}})\frac{k}{ku}+\mathcal{P}F_{,\mathcal{P}}\frac{\dot{k}}{\dot{k}u}\right),$$
$$\Pi=\frac{m\ell}{ku}\left(F_{,\mathcal{P}}u-2\mathcal{P}F_{,\mathcal{Q}}\frac{\dot{k}}{u\dot{k}}\right),$$

where $u\equiv\dot{x}/\sqrt{\dot{x}\dot{x}}$ and the coma sign stands for differentiation. As a consequence of relativistic invariance of action (2.1), the identity $0=\int_{\tau_1}^{\tau_2}\delta L=-(P\delta x+\Pi\delta k)_{\tau_2}+(P\delta x+\Pi\delta k)_{\tau_1}$ holds for infinitesimal Poincaré transformations of solutions[1]. The invariance with respect to space-time translations, $\delta x^\mu=\epsilon^\mu=$ const. and $\delta k^\mu=0$, implies conservation of momentum $P^\mu$, whereas the invariance with respect to space-time rotations, $\delta x^\mu=\Omega^\mu{}_\nu x^\nu$ and $\delta k^\mu=\Omega^\mu{}_\nu k^\nu$ ($\Omega_{\mu\nu}=$ const. and $\Omega_{(\mu\nu)}=0$), implies conservation of angular momentum $M_{\mu\nu}\equiv x_\mu P_\nu-x_\nu P_\mu+k_\mu\Pi_\nu-k_\nu\Pi_\mu$. These constants of motion are used to form both of the Casimir invariants of the Poincaré group: $P_\mu P^\mu$ and $W_\mu W^\mu$, where $W^\mu$ is the Pauli–Lubański (space-like) spin-pseudovector: $W^\mu=-\frac{1}{2}\epsilon^{\mu\alpha\beta\gamma}M_{\alpha\beta}P_\gamma$, hence[2]

$$PP=m^2\big((F-\mathcal{P}F_{,\mathcal{P}})(F-\mathcal{P}F_{,\mathcal{P}}-4\mathcal{Q}F_{,\mathcal{Q}})-\mathcal{Q}F_{,\mathcal{P}}{}^2\big),$$
$$WW=-m^4\ell^2\mathcal{Q}\big(F_{,\mathcal{P}}{}^2+2F_{,\mathcal{Q}}(F-\mathcal{P}F_{,\mathcal{P}})\big)^2.$$

The function $F$ should not be chosen at random, but be determined uniquely based on some clear-cut physical idea. In this respect fundamental conditions are imposed. The unspecified arbitrary parameters $m$ and $\ell$ can now be set by relating them directly to the fixed numerical values of the Casimir invariants. With no loss to generality, this can be done by requiring that

$$PP\equiv m^2,\qquad WW\equiv-\tfrac{1}{4}m^4\ell^2. \tag{2.2}$$

Unfortunately, similar to the fundamental relativistic rotator, fundamental breathing rotators turn out to be defective as dynamical systems due to vanishing of the Hessian determinant. Here, by a Hessian determinant this is understood to be the determinant of a matrix of second derivatives of a Lagrangian with respect to velocities associated only with the dynamical degrees of freedom.

A quick overview of this is best understood by the astounding relationship of the Hessian determinant (denoted by $\det\mathcal{H}$) with a Jacobian determinant of the following $F$-dependent mapping, leading from coordinates $(\mathcal{P},\mathcal{Q})$ to coordinates $(PP,WW)$ (cf appendix B for a derivation)

$$\det\mathcal{H}=\kappa\cdot\frac{F-\mathcal{P}F_{,\mathcal{P}}}{F_{,\mathcal{P}}(\mathcal{P}^2+\mathcal{Q})-\mathcal{P}F}\cdot\left|\frac{\partial(PP,WW)}{\partial(\mathcal{P},\mathcal{Q})}\right|. \tag{2.3}$$

Here, $\kappa$ is some kinematical factor, the same for all $F$. In the distinguished case, when $F=\sqrt{1+\frac{\mathcal{P}^2}{\mathcal{Q}}}S(\mathcal{Q})$ (then $\partial_\mathcal{P}F(\mathcal{P}^2+\mathcal{Q})-\mathcal{P}F=0$), the Jacobian determinant vanishes, but this

[1] For the purpose of this section it suffices to keep in mind that $kk=0$. However, in order to find equations of motion in a covariant form, one must add to the Hamilton's action an appropriate term with a Lagrange multiplier.

[2] $WW=-\begin{vmatrix}PP & Pk & P\Pi\\ kP & kk & k\Pi\\ \Pi P & \Pi k & \Pi\Pi\end{vmatrix}$.

is not necessarily the case for the Hessian determinant (indeterminate form $\frac{0}{0}$). Indeed, then the Casimir invariants are functionally dependent: $PP = m^2 S(S - 4\mathcal{Q}\partial_{\mathcal{Q}} S)$ and $WW = -(2m^2 \ell S\sqrt{\mathcal{Q}}\partial_{\mathcal{Q}} S)^2$, while $\det\mathcal{H} \propto \frac{\mathcal{Q}S^3\partial_{\mathcal{Q}}S}{(\mathcal{P}^2+\mathcal{Q})^2}(2\mathcal{Q}(\partial_{\mathcal{Q}}S)^2 + S(\partial_{\mathcal{Q}}S + 2\mathcal{Q}\partial_{\mathcal{Q}\mathcal{Q}}S))$, that is, $\det\mathcal{H} \propto S^3\partial_{\mathcal{Q}}S\partial_{\mathcal{Q}}(PP) \propto S^2\partial_{\mathcal{Q}}(WW)$, which is nonzero unless fundamental conditions are imposed. In all other cases, when $\partial_{\mathcal{P}}F(\mathcal{P}^2 + \mathcal{Q}) - \mathcal{P}F \neq 0$, vanishing of the Hessian determinant is equivalent to vanishing of the Jacobian determinant (if $F - \mathcal{P}\partial_{\mathcal{P}}F = 0$ then $WW = m^2\ell^2 PP$, which is unphysical as then $PP < 0$ because pseudovector $W$, being orthogonal to a null vector, is always space-like).

## 3. Conclusions

In this paper a class of relativistic dynamical systems, described by a single null vector associated with a space-time position and defined by a relativistically invariant action (2.1), has been examined. To distinguish them from the class of rotators introduced by Staruszkiewicz [1], the systems are called 'breathing rotators'. Breathing rotators have six dynamical degrees of freedom—three for position, two for the null direction associated with null vector $k$ and one 'breathing' degree of freedom associated with the amplitude of $k$. As usual in relativity theory, Lagrangians must be reparametrization invariant; thus the arbitrary parameter $\tau$ is not dynamical and is treated as a gauge variable.

There are two subclasses of breathing rotators that are distinguished by analytical properties of relation (2.3). Rotators with singular Hessian form a subset of breathing rotators with functionally dependent Casimir invariants $PP$ and $WW$, whereas functional independence of the invariants guarantees non-singularity of the Hessian.

Another result is that breathing rotators that are fundamental have singular Hessian. This property makes them defective as dynamical systems. So far, this has been the second example of relativistically invariant systems for which fundamental conditions imply singularity of the Hessian. The other is provided by the Staruszkiewicz fundamental rotator [1], or, equivalently, the $(m, s)$ particle of Kuzenko *et al* [6], for which this property has been discovered in [5]. However, it seems rather improbable that fundamental conditions would always imply singularity of the Hessian. Also, a proof of such a theorem seems hopelessly difficult in a generic situation. Therefore, it is necessary in the future to construct a counterexample.

As follows from appendix A, two breathing fundamental rotators are possible with the following Hamilton's actions

$$S = -m\int \mathrm{d}\tau\sqrt{\dot{x}\dot{x}}\sqrt{\left[1 - \frac{(\dot{k}\dot{x})(\dot{k}\dot{x})}{(\dot{x}\dot{x})(\dot{k}\dot{k})}\right]\left[1 \pm \sqrt{-\ell^2\frac{\dot{k}\dot{k}}{(k\dot{x})^2}}\right]}, \tag{3.1}$$

$$S_\nu = -m\int \mathrm{d}\tau\sqrt{\dot{x}\dot{x}}\left(\sqrt{1 \pm \sqrt{-\ell^2\frac{\dot{k}\dot{k}}{(k\dot{x})^2}} + \nu^2\ell^2\frac{\dot{k}\dot{k}}{(k\dot{x})^2}} + \nu\ell\frac{\dot{k}\dot{x}}{k\dot{x}\sqrt{\dot{x}\dot{x}}}\right), \qquad \nu \in \mathbb{R}. \tag{3.2}$$

The parameter $\nu$ is an integration constant of fundamental conditions, and can be reinterpreted as an additional length scale: $\nu\ell$. It should be stressed here that the Casimir invariants for action (3.2) are independent of $\nu$. For both rotators

$$PP = m^2, \qquad WW = -\tfrac{1}{4}m^4\ell^2.$$

Contrary to rotator (3.1) which has six degrees of freedom, rotator (3.2) must be treated as having only five degrees of freedom, since the amplitude of $k$ in this case is a gauge variable. Indeed, for any function $\psi(\tau)$

$$S_{\nu}[x, e^{\psi}k] = S_{\nu}[x, k] - m\ell\nu\psi(\tau).$$

Since the corresponding Lagrangians differ by a total derivative, the form of equations of motion is left unchanged. This means that the breathing mode separates completely from the dynamics of the other degrees of freedom and does not influence them at all, therefore it can be completely ignored. As a result, the dynamical system defined by action (3.2) depends on position and a null direction only, similar to the fundamental relativistic rotator. Unfortunately, rotator (3.2) cannot be considered as a replacement for the fundamental relativistic rotator, which is obtained by setting $\nu = 0$,

$$S_{\nu=0} = -m\int d\tau\sqrt{\dot{x}\dot{x}}\sqrt{1+\sqrt{-\ell^2\frac{\dot{k}\dot{k}}{(k\dot{x})^2}}},$$

since the determinant of a reduced Hessian matrix corresponding to the five dynamical degrees of freedom of rotator (3.2) (the breathing mode is excluded), vanishes as well. Summing up, these are the fundamental conditions that are responsible for the singular behavior of both rotators (3.1) and (3.2).

## Appendix A. Solution of fundamental conditions

To solve fundamental conditions (2.2), it is convenient to recast them into the equivalent form

$$\left.\begin{aligned} 4u^2 - 4u(1 + xu_x + yu_y) + 2xyu_xu_y + (y^2 - x^2){u_y}^2 = 0\\ 2u + 2uu_x - yu_xu_y + x{u_y}^2 = 0 \end{aligned}\right\},$$

where $x \equiv \pm\sqrt{\mathcal{Q}}$, $y \equiv \mathcal{P}$ and $\sqrt{u(x, y)} \equiv F(\mathcal{P}, \mathcal{Q})$, $u > 0$. The second equation can be linearized by the Legendre transformation $u(x, y) \to x\xi + y\eta - \omega(\xi, \eta)$, $x \to \omega_{\xi}$, $y \to \omega_{\eta}$, $u_x \to \xi$, $u_y \to \eta$ going over into $\eta(2 + \xi)\omega_{\eta} + (\eta^2 + 2\,\xi(1 + \xi))\omega_{\xi} = 2(1 + \xi)\omega$. The inhomogeneous term is removed by the substitution $\omega(\xi, \eta) = (\eta^2 + \xi^2)h(\xi, \eta)/(\eta^2 - 2\,\xi)$, giving the equivalent equation $\eta(2+\xi)h_{\eta} + (\eta^2 + 2\,\xi(1+\xi))\,h_{\xi} = 0$ which is solved by noting that gradient $\{h_{\xi}, h_{\eta}\}$ must be collinear with vector $\{\eta(2+\xi), -\eta^2 - 2\,\xi(1+\xi)\}$. The latter is proportional to a gradient of any function of a single argument $s$, $s \equiv \sqrt{\eta^2 + \xi^2}/(\eta^2 - 2\,\xi)$. Hence, the general solution for $\omega$ must be of the form $\omega(\xi, \eta) = s\,g(s)\sqrt{\eta^2 + \xi^2}$ with arbitrary function $g$.

By means of the same Legendre transformation and with the obtained ansatz for $\omega$, the first equation goes over into

$$\eta^2 s^2[4(1 - s^2)g^2 + 4g(1 + (1 - 2s^2)sg') + sg'(4 + (1 - 4s^2)sg')] = 0.$$

It is remarkable, that all coefficients in this equation can be expressed by $s$ alone, showing that two quite distinct equations of different physical origin, namely the fundamental conditions, have a common solution. The terms linear in $g$ and $g'$ can be absorbed by introduction of function $g(s) \equiv G(s) - (2s^2)^{-1}$, and next, the term proportional to $GG'$ can be absorbed with the help of function $f(s)$ defined as $s^2\,G(s) \equiv \sqrt{1 - 4s^2}\,f(s)$. One is now only left with the simple equation $1 = 4\,f^2 - (1 - 4\,s^2)^2(f')^2$. Its first, trivial solution, is $f(s) = \pm 1/2$. The other solution can be found by substitution $f(s) = \pm\cosh(\psi(s))/2$, hence $((1 - 4s^2)\psi'(s))^2 = 4$, which is easily integrable and has an integration constant $\alpha$.

Finally, the two resulting solutions for $g$ are: $g(s) = (-1 \pm \sqrt{1-4s^2})/(2s^2)$ and $g(s) = (-1 \pm \cosh(\alpha) + 2s\,\sinh(\alpha))/(2s^2)$. The corresponding solutions for $\omega$ are $\omega(\xi,\eta) = \xi - \eta(\eta \pm \sqrt{\eta^2 - 4(1+\xi)})/2$ and $\omega(\xi,\eta) = (\eta^2 - 2\xi)(-1 \pm \cosh(\alpha))/2 + \sqrt{\eta^2+\xi^2}\,\sinh(\alpha)$, respectively, the latter with a real parameter $\alpha$. The last point is to apply the inverse Legendre transformation to obtain the corresponding solutions for $u(x,y)$ and thence for $F(\mathcal{P},\mathcal{Q})$.

Finally, there are two solutions of fundamental conditions (2.2)

$$F(\mathcal{P},\mathcal{Q}) = \pm\sqrt{(1\pm\sqrt{\mathcal{Q}})\left(1+\frac{\mathcal{P}^2}{\mathcal{Q}}\right)},$$

$$F(\mathcal{P},\mathcal{Q}) = \frac{1}{a}\left(\mathcal{P} \pm \sqrt{(1\pm\sqrt{\mathcal{Q}})a^2 - \mathcal{Q}}\right), \qquad a \in \mathbb{R},$$

where $a = 2\sinh(\alpha/2)$ (repeated $\pm$ signs in a solution are not related to each other). A formal limit $a \to \infty$ reproduces the original Lagrangian of the fundamental relativistic rotator.

## Appendix B. Hessian determinant for breathing rotators

Consider the determinant of a matrix of second derivatives of the Lagrangian in action (2.1) with respect to generalized velocities, associated with the physical degrees of freedom only. Prior to calculation of it, a convenient map of internal coordinates is to be chosen, and any spurious degrees of freedom eliminated (fixing 'gauge'). The result is always a factor of the same invariant—a second order differential operator with respect to the Lorentz invariants $\mathcal{P}$ and $\mathcal{Q}$ and acting on function $F$—times an unimportant geometrical factor dependent on the particular map chosen. For the purpose of this paper, this differential invariant (up to a constant factor) is called the Hessian determinant.

To calculate it, one must fix the arbitrary parameter $\tau$. Let it be the time coordinate in a given inertial coordinate frame, times a constant dimensional factor, $\tau \equiv t = \ell^{-1}x^0$. One may chose a map of internal coordinates in which $\dot{x} = \ell[1,\mathbf{V}^{\mathbf{T}}]$, $k = \mathrm{e}^{\Psi}[1,\mathbf{N}^{\mathbf{T}}]$ with $\mathbf{N}^{\mathbf{T}}\mathbf{N} = 1$, $\dot{k} = \mathrm{e}^{\Psi}[\mathbf{N}^{\mathbf{T}}\mathbf{\Omega},\mathbf{\Omega}^{\mathbf{T}}]$ and $\mathbf{\Omega} \equiv \dot{\mathbf{N}} + \dot{\Psi}\mathbf{N}$ ($\dot{\Psi} \equiv \mathbf{N}^{\mathbf{T}}\mathbf{\Omega}$), where $\mathbf{V}$ and $\mathbf{\Omega}$ stand for the six generalized velocities associated with position $x$ and null vector $k$.[3] Up to a constant factor, the Lagrangian in action (2.1) is given by a dimensionless scalar $\mathcal{L}$

$$\mathcal{L} = -\sqrt{1-\mathbf{V}^{\mathbf{T}}\mathbf{V}}\,F(\mathcal{P},\mathcal{Q}), \qquad \mathcal{P} = \frac{1}{\sqrt{1-\mathbf{V}^{\mathbf{T}}\mathbf{V}}}\frac{\mathbf{N}^{\mathbf{T}}\mathbf{\Omega} - \mathbf{V}^{\mathbf{T}}\mathbf{\Omega}}{1-\mathbf{N}^{\mathbf{T}}\mathbf{V}}, \qquad \mathcal{Q} = \frac{\mathbf{\Omega}^{\mathbf{T}}\mathbf{\Omega} - (\mathbf{N}^{\mathbf{T}}\mathbf{\Omega})^2}{(1-\mathbf{N}^{\mathbf{T}}\mathbf{V})^2}.$$

The $6\times6$ square matrix $\mathcal{H}$ of second derivatives of $\mathcal{L}$ with respect to three-velocities represented by column vectors $\mathbf{V}$ and $\mathbf{\Omega}$, has a block structure

$$\mathcal{H} = \begin{bmatrix} \mathcal{L}_{\mathbf{V}\mathbf{V}^{\mathbf{T}}} & \mathcal{L}_{\mathbf{V}\mathbf{\Omega}^{\mathbf{T}}} \\ (\mathcal{L}_{\mathbf{V}\mathbf{\Omega}^{\mathbf{T}}})^T & \mathcal{L}_{\mathbf{\Omega}\mathbf{\Omega}^{\mathbf{T}}} \end{bmatrix},$$

with elements being matrices of size $3\times3$, explicitly given below ($\gamma^{-1} = \sqrt{1-\mathbf{V}^{\mathbf{T}}\mathbf{V}}$, $\chi^{-1} = 1-\mathbf{N}^{\mathbf{T}}\mathbf{V}$, $\zeta = \gamma\chi\mathbf{V}^{\mathbf{T}}\mathbf{\Omega}$),

$$\begin{aligned}\mathcal{L}_{\mathbf{V}\mathbf{V}^{\mathbf{T}}} = {} & (F - \mathcal{P}F_{,\mathcal{P}})\gamma\mathbf{E} + \gamma^3(F - \mathcal{P}(F_{,\mathcal{P}} + \mathcal{P}F_{,\mathcal{P}\mathcal{P}}))\mathbf{V}\mathbf{V}^{\mathbf{T}} - \gamma\chi^2F_{,\mathcal{P}\mathcal{P}}\mathbf{\Omega}\mathbf{\Omega}^{\mathbf{T}} \\ & - \gamma^{-1}\chi^2(\mathcal{P}(2F_{,\mathcal{P}} + \mathcal{P}F_{,\mathcal{P}\mathcal{P}}) + 2\mathcal{Q}(3F_{,\mathcal{Q}} + 2(\mathcal{P}F_{,\mathcal{P}\mathcal{Q}} + \mathcal{Q}F_{,\mathcal{Q}\mathcal{Q}})))\mathbf{N}\mathbf{N}^{\mathbf{T}} \\ & - \gamma\chi(\mathcal{P}^2F_{,\mathcal{P}\mathcal{P}} + 2\mathcal{Q}(\mathcal{P}F_{,\mathcal{P}\mathcal{Q}} - F_{,\mathcal{Q}}))(\mathbf{N}\mathbf{V}^{\mathbf{T}} + \mathbf{V}\mathbf{N}^{\mathbf{T}})\ldots\end{aligned}$$

[3] Here, bold capitals stand for column three-vectors, then $\mathbf{X}^{\mathbf{T}}\mathbf{Y} \equiv \mathbf{Y}^{\mathbf{T}}\mathbf{X}$ is the scalar product of vectors $\mathbf{X}$ and $\mathbf{Y}$, while $\mathbf{X}\mathbf{Y}^{\mathbf{T}}$ is a $3\times3$ matrix, in general, distinct from $\mathbf{Y}\mathbf{X}^{\mathbf{T}}$.

$$
\begin{aligned}
&+\ \chi^2(F_{,\mathcal{P}}+\mathcal{P}F_{,\mathcal{PP}}+2\mathcal{Q}F_{,\mathcal{PQ}})(\mathbf{N\Omega^T}+\mathbf{\Omega N^T})+\gamma^2\chi\mathcal{P}F_{,\mathcal{PP}}(\mathbf{V\Omega^T}+\mathbf{\Omega V^T}),\\
\mathcal{L}_{\mathbf{V\Omega^T}} =\ &\chi F_{,\mathcal{P}}\mathbf{E}+2\chi^3F_{,\mathcal{PQ}}\mathbf{\Omega\Omega^T}+\gamma^2\chi\mathcal{P}F_{,\mathcal{PP}}\mathbf{VV^T}\\
&+\ \chi^2(\gamma)^{-2}(2(\mathcal{P}+\zeta)(\mathcal{P}F_{,\mathcal{PQ}}+2(F_{,\mathcal{Q}}+\mathcal{Q}F_{,\mathcal{QQ}}))-\gamma^2(F_{,\mathcal{P}}+\mathcal{P}F_{,\mathcal{PP}}+2\mathcal{Q}F_{,\mathcal{PQ}}))\mathbf{NN^T}\\
&+\chi^2(F_{,\mathcal{P}}+\mathcal{P}F_{,\mathcal{PP}}+2\mathcal{Q}F_{,\mathcal{PQ}})\mathbf{NV^T}-2\gamma^{-1}\chi^3(\mathcal{P}F_{,\mathcal{PQ}}+2(F_{,\mathcal{Q}}+\mathcal{Q}F_{,\mathcal{QQ}}))\mathbf{N\Omega^T}\\
&+\chi(2(\mathcal{P}+\zeta)(\mathcal{P}F_{,\mathcal{PQ}}-F_{,\mathcal{Q}})-\gamma^2\mathcal{P}F_{,\mathcal{PP}})\mathbf{VN^T}-\gamma\chi^2F_{,\mathcal{PP}}\mathbf{\Omega V^T}\\
&-2\gamma\chi^2(\mathcal{P}F_{,\mathcal{PQ}}-F_{,\mathcal{Q}})\mathbf{V\Omega^T}+\gamma^{-1}\chi^2(\gamma^2F_{,\mathcal{PP}}-2(\mathcal{P}+\zeta)F_{,\mathcal{PQ}})\mathbf{\Omega}N^{\mathbf{T}},\\
\mathcal{L}_{\mathbf{\Omega\Omega}^T} =\ &-2\gamma^{-1}\chi^2F_{,\mathcal{Q}}\mathbf{E}-\gamma\chi^2F_{,\mathcal{PP}}\mathbf{VV^T}-4\gamma^{-1}\chi^4F_{,\mathcal{QQ}}\mathbf{\Omega\Omega^T}\\
&+\ \gamma^{-3}\chi^2(\gamma^2(2F_{,\mathcal{Q}}+4(\mathcal{P}+\zeta)F_{,\mathcal{PQ}}-\gamma^2F_{,\mathcal{PP}})-4(\mathcal{P}+\zeta)^2F_{,\mathcal{QQ}})\mathbf{NN^T}\\
&+\gamma^{-1}\chi^2(\gamma^2F_{,\mathcal{PP}}-2(\mathcal{P}+\zeta)F_{,\mathcal{PQ}})(\mathbf{NV^T}+\mathbf{VN^T})+2\chi^3F_{,\mathcal{PQ}}(\mathbf{V\Omega^T}+\mathbf{\Omega V^T})\\
&+2\gamma^{-2}\chi^3(2(\mathcal{P}+\zeta)F_{,\mathcal{QQ}}-\gamma^2F_{,\mathcal{PQ}})(\mathbf{N\Omega^T}+\mathbf{\Omega N^T}).
\end{aligned}
$$

Above matrices are linear combinations of a $3\times 3$ unit matrix $\mathbf{E}$ and *nine* elementary matrices $\mathbf{NN^T}$, $\mathbf{NV^T}$, $\mathbf{N\Omega^T}$, $\mathbf{VN^T}$, $\mathbf{VV^T}$, $\mathbf{V\Omega^T}$, $\mathbf{\Omega N^T}$, $\mathbf{\Omega V^T}$, $\mathbf{\Omega\Omega^T}$ all of size $3\times 3$. Sums, products, inverses and transpositions leave this structure invariant. Up to a factor, all matrices with the same structure can be written as $\mathbf{E}+\mathbf{NX^T}+\mathbf{VY^T}+\mathbf{\Omega}Z^{\mathbf{T}}$, where $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are linear combinations of $\mathbf{N}$, $\mathbf{V}$, $\mathbf{\Omega}$. This trivial observation enables to use the following identity for their determinants

$$
\det(\mathbf{E}+\mathbf{NX^T}+\mathbf{VY^T}+\mathbf{\Omega Z^T}) = \begin{vmatrix} 1+\mathbf{N^TX} & \mathbf{N^TY} & \mathbf{N^TZ}\\ \mathbf{V^TX} & 1+\mathbf{V^TY} & \mathbf{V^TZ}\\ \mathbf{\Omega^TX} & \mathbf{\Omega^TY} & 1+\mathbf{\Omega^TZ}\end{vmatrix}.
$$

Now, on account of the important identity

$$
\det\mathcal{H} = \det(\mathcal{L}_{\mathbf{\Omega\Omega^T}})\det(\mathcal{L}_{\mathbf{VV^T}}-\mathcal{L}_{\mathbf{V\Omega^T}}(\mathcal{L}_{\mathbf{\Omega\Omega^T}})^{-1}(\mathcal{L}_{\mathbf{V\Omega^T}})^T),
$$

the task of computing the Hessian determinant simplifies significantly. The only thing left is to calculate the inverse of $\mathcal{L}_{\mathbf{\Omega\Omega}^T}$, which is also easy, since it can be found by solving the linear system of equations $\left(\mathcal{L}_{\mathbf{\Omega\Omega^T}}\right)^{-1}\left(\mathcal{L}_{\mathbf{\Omega\Omega^T}}\right)=\mathbf{E}$ for ten unknown expansion coefficients of $\left(\mathcal{L}_{\mathbf{\Omega\Omega^T}}\right)^{-1}$ in the base of elementary matrices (to carry out these calculations one may assume that the Gramian determinant for the scalar products of vectors $\mathbf{N}$, $\mathbf{V}$, $\mathbf{\Omega}$ is nonzero). Equipped with this knowledge, one can show, after a lengthy and tedious but straightforward calculation, that

$$
\begin{aligned}
\det\mathcal{H} = &-\frac{(F-\mathcal{P}F_{,\mathcal{P}})(F_{,\mathcal{P}}{}^2+2F_{,\mathcal{Q}}(F-\mathcal{P}F_{,\mathcal{P}}))}{(1-\mathbf{N^TV})^4(1-\mathbf{V^TV})^2}\times\cdots\\
&\cdots\left(\left(F_{,\mathcal{P}}{}^2+F_{,\sqrt{\mathcal{Q}}}{}^2\right)F_{,\mathcal{PP}}+(F-\mathcal{P}F_{,\mathcal{P}})\left|\frac{\partial(F_{,\mathcal{P}},F_{,\sqrt{\mathcal{Q}}})}{\partial(\mathcal{P},\sqrt{\mathcal{Q}})}\right|\right).
\end{aligned}
$$

For comparison, it is interesting to calculate the Jacobian determinant of a mapping $(\mathcal{P},\mathcal{Q})\to(PP(\mathcal{P},\mathcal{Q}),WW(\mathcal{P},\mathcal{Q}))$

$$
\begin{aligned}
\frac{\left|\frac{\partial(PP,WW)}{\partial(\mathcal{P},\mathcal{Q})}\right|}{-2m^6\lambda^2(F_{,\mathcal{P}}(\mathcal{P}^2+\mathcal{Q})-\mathcal{P}F)} &= \left(F_{,\mathcal{P}}{}^2+2F_{,\mathcal{Q}}(F-\mathcal{P}F_{,\mathcal{P}})\right)\times\cdots\\
&\cdots\left(\left(F_{,\mathcal{P}}{}^2+F_{,\sqrt{\mathcal{Q}}}{}^2\right)F_{,\mathcal{PP}}+(F-\mathcal{P}F_{,\mathcal{P}})\left|\frac{\partial(F_{,\mathcal{P}},F_{,\sqrt{\mathcal{Q}}})}{\partial(\mathcal{P},\sqrt{\mathcal{Q}})}\right|\right).
\end{aligned}
$$

The Jacobian is proportional to $\det\mathcal{H}$, at least if $F_{,\mathcal{P}}(\mathcal{P}^2+\mathcal{Q})-\mathcal{P}\,F\neq 0$.

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